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IR finite one-loop box scalar integral with massless internal lines

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Abstract. The IR finite one-loop box scalar integral with massless internal lines has been recalculated. The result is very compact, simple and valid for arbitrary values of the relevant kinematic variables. It is given in terms of only two dilogarithms and a few logarithms, all of very simple arguments.

1 Introduction

Exclusive hadronic processes at large momentum transfer in which the total of particles (partons) in the initial and final states is $N \geq 5$ are becoming increasingly important in testing various aspects of QCD. The tree-level results for the cross section of these processes, being proportional to a high power of the QCD coupling constant α_s^N, are very sensitive to the variation of the renormalization scale and scheme being used. Consequently, to stabilize the leading-order (tree-level) predictions and to achieve a complete confrontation between theoretical predictions and experimental data, at least one-loop (NLO) corrections are necessary.

The main theoretical difficulty in obtaining the NLO predictions consists in the treatment of the occurring N-point one-loop scalar and tensor integrals with massless internal lines. Through scalarization and reduction procedures [1–3], the computation of such integrals reduces to the calculation of a set comprised of six box ($N = 4$) diagrams, one of which is finite while the other five are IR divergent. Being fundamental for one-loop calculations in pQCD with massless quarks, it is very important for practical purposes that the results for these integrals are obtained in forms as compact as possible (expressed in terms of as few as possible dilogarithmic functions) with as simple as possible arguments.

The IR divergent basic box integrals have been considered in [4–6]. A more general and detailed analysis has been given in [7]. As for the IR finite box integral, it has been calculated in [8], and later obtained in a more compact form in [9]. From the practical point of view, in addition to not being most compact, a disadvantage of this result is a very complicated structure of the arguments of the occurring dilogarithmic and logarithmic functions. An alternative expression for the IR finite box integral has been obtained in [10], where it has been related to the IR finite triangle scalar integral. The result of [10] expressed in terms of only two dilogarithms, however, is not valid for arbitrary values of the relevant kinematic variables.

Making use of the method of [10], in this work we carefully recalculate the IR finite box scalar integral. Our result is as simple and compact as that of [10] and, as the result of [8,9] is quite general, i.e. applicable to all kinematic regions. In addition to being more compact and general, the advantage of our result over the results previously obtained is also the fact that one can very easily separate the real and imaginary parts of logarithms and dilogarithms, making it more appropriate for numerical calculations.

This paper is organized as follows. Section 2 is devoted to introducing the notation and to some preliminary considerations. In Sect. 3, using the Feynman parameter method and the Mellin–Barnes integral representations, we evaluate the IR finite box scalar integral by relating it to the IR finite triangle scalar integral. In Sect. 4 we give some concluding remarks. The details of the evaluation of the IR finite triangle scalar integral are given in the appendix.

2 Preliminaries

The scalar one-loop box integral with massless internal lines in D-dimensional space-time is given by

$$I_4(p_1, p_2, p_3, p_4) = (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{A_1 A_2 A_3 A_4}, \tag{1}$$

where p_i ($i = 1, 2, 3, 4$) are the external momenta, l is the loop momentum and μ is the usual dimensional regularization scale. As indicated in Fig. 1, all external momenta are taken to be incoming, so that the massless propagators are

$$\begin{aligned} A_1 &= l^2 + \mathrm{i}\epsilon, \\ A_2 &= (l + p_1)^2 + \mathrm{i}\epsilon, \\ A_3 &= (l + p_1 + p_2)^2 + \mathrm{i}\epsilon, \\ A_4 &= (l + p_1 + p_2 + p_3)^2 + \mathrm{i}\epsilon, \end{aligned} \tag{2}$$

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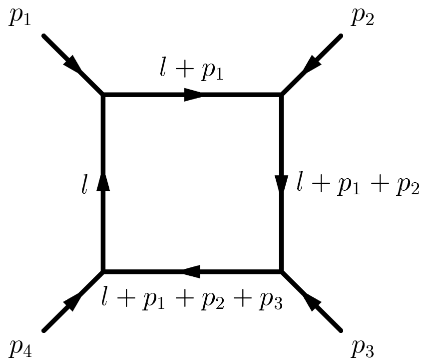

Fig. 1. One-loop box diagram

where the quantity $\mathrm{i}\epsilon$ ($\epsilon > 0$), represents an infinitesimal imaginary part specifying on which side of the cut a multivalued function has to be evaluated. We take the cut along the negative real axis.

Combining the denominators using the Feynman parametrization formula, performing the D-dimensional loop momentum integration, introducing the external "masses" $p_i^2 = m_i^2$ ($i = 1, 2, 3, 4$) and the Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$, the integral (1) becomes

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\frac{\Gamma(4-D/2)}{(4\pi\mu^2)^{D/2-2}} \times \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\ \delta(x_1+x_2+x_3+x_4-1) \times \left(-x_1x_3\ s - x_2x_4\ t - x_1x_2\ m_1^2 -x_2x_3\ m_2^2 - x_3x_4\ m_3^2 - x_1x_4\ m_4^2 - \mathrm{i}\epsilon\right)^{D/2-4}. \quad (3)$$

Depending on the number of external massless lines, there are six special cases of the above integral. Adopting the notation of [6], we denote them by

$$I_4^{0m} \equiv I_4(s,t;0,0,0,0,), \quad (4)$$
$$I_4^{1m} \equiv I_4(s,t;0,0,0,m_4^2), \quad (5)$$
$$I_4^{2me} \equiv I_4(s,t;0,m_2^2,0,m_4^2), \quad (6)$$
$$I_4^{2mh} \equiv I_4(s,t;0,0,m_3^2,m_4^2), \quad (7)$$
$$I_4^{3m} \equiv I_4(s,t;0,m_2^2,m_3^2,m_4^2), \quad (8)$$
$$I_4^{4m} \equiv I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2). \quad (9)$$

The above integrals constitute the fundamental set of integrals, in the sense that an arbitrary one-loop $N(\geq 5)$-point integral with massless internal lines can be uniquely expressed as a linear combination of these integrals with the coefficients being rational functions of the relevant kinematic variables. The integrals (4)–(8) are IR divergent and need to be evaluated in $D = 4+2\varepsilon_{\mathrm{IR}}$ ($\varepsilon_{\mathrm{IR}} > 0$) dimensions [7]. On the other hand, the integral (9) is IR finite and can be calculated in $D = 4$ dimensions.

Setting $D = 4$ in (3), we obtain

$$I_4(s,t,m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \times \delta(x_1+x_2+x_3+x_4-1)\left(x_1x_3\ s + x_2x_4\ t + x_1x_2\ m_1^2 +\ x_2x_3\ m_2^2 + x_3x_4\ m_3^2 + x_1x_4\ m_4^2 + \mathrm{i}\epsilon\right)^{-2}. \quad (10)$$

This integral is invariant under some permutations of external "masses" and Mandelstam variables. Namely, introducing the following set of ordered pairs

$$\{(s,t),(m_1^2,m_3^2),(m_2^2,m_4^2)\}, \quad (11)$$

one can easily see that the integral (10) is invariant under the permutations of ordered pairs given in (11), as well as under the simultaneous exchange of places of elements in any two pairs. Furthermore, the integral is invariant under the simultaneous change of the signs of the kinematic variables and the causal $\mathrm{i}\epsilon$. Consequently, it is sufficient to analyze only the cases of the integral (10) in which the number of positive kinematic variables is larger than the number of negative ones.

3 Calculation and results

Eliminating the δ-function in (10) by performing the x_4 integration, we get

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\int_0^1 \mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2\int_0^{1-x_1-x_2}\mathrm{d}x_3 \times \Big[(1-x_1-x_2-x_3)(x_1m_4^2+x_2t+x_3m_3^2) + x_1x_2m_1^2 + x_1x_3s + x_2x_3m_2^2 + \mathrm{i}\epsilon\Big]^{-2}. \quad (12)$$

Next, if we first make the substitution $x_3 \to (1-x_1-x_2)x_3$ and then $x_2 \to (1-x_1)x_2$, we find that

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2) \quad (13)$$
$$= \frac{\mathrm{i}}{(4\pi)^2}\int_0^1 \mathrm{d}x_1\int_0^1 \mathrm{d}x_2\int_0^1 \mathrm{d}x_3 \times (1-x_2)\Big\{(1-x_1)(1-x_2) \times \left[x_2x_3m_2^2 + x_2(1-x_3)t + (1-x_2)x_3(1-x_3)m_3^2\right] +x_1\left[x_2m_1^2 + (1-x_2)x_3s + (1-x_2)(1-x_3)m_4^2\right]+\mathrm{i}\epsilon\Big\}^{-2}.$$

Performing the x_1 integration and a few simple rearrangements, the integral takes the form

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mathrm{i}}{(4\pi)^2}\int_0^1 \mathrm{d}x_2\int_0^1 \mathrm{d}x_3 \times \left[x_2m_1^2 + (1-x_2)x_3s + (1-x_2)(1-x_3)m_4^2 + \mathrm{i}\epsilon\right]^{-1} \times \Big[x_2x_3m_2^2 + x_2(1-x_3)t + (1-x_2)x_3(1-x_3)m_3^2 + \mathrm{i}\epsilon\Big]^{-1}. \quad (14)$$

Next, doing the x_2 integration and expressing the remaining integral in terms of logarithms and dilogarithms, one arrives at the result obtained in [9], which is valid for all values of the kinematic variables. The result of [9] is given in terms of logarithmic and dilogarithmic functions with the argument of the form

$$1+\frac{f_1-\mathrm{i}\epsilon}{f_2-\mathrm{i}\epsilon}\left(f_3\pm\sqrt{f_4+\mathrm{i}f_5\epsilon}\right),$$

where f_i $(i=1,2,3,4,5)$ are real functions of the kinematic variables. As seen from the above, the structure of the argument is very complicated since iϵ appears both outside and inside the square root. There are two consequences of this: first, it is very difficult to determine the sign of the infinitesimal imaginary part of the argument; second, it is extremely complicated to use the dilogarithmic identities with the aim of simplifying the final result. It should also be pointed out that since the final result contains products of logarithms, the signs of the imaginary parts of arguments are important not only for determining the imaginary part of the result, but also for the real part of the result. From the practical point of view, this represents a disadvantage of the result for the IR finite massless box scalar integral derived in [9].

To avoid the above-mentioned problems, we take a different approach and proceed to evaluate the box integral by expressing it using the Mellin–Barnes integral representation [10]. To this end, we first need to find the Mellin–Barnes representation of the following expression:

$$J=(y_1y_2a+y_1(1-y_2)b+(1-y_1)y_2(1-y_2)c+\mathrm{i}\epsilon)^{-1}, \quad (15)$$

where $y_1, y_2\in[0,1]$, and a, b and c are arbitrary real variables different from zero. In what follows we also need the Mellin–Barnes representation of the expression

$$(1-z)^{-k}=\frac{1}{2\pi\mathrm{i}\Gamma(k)}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{d}s\Gamma(s)\Gamma(k-s)(-z)^{-s},$$
$$0<\gamma<\mathrm{Re}k, \quad |\arg(-z)|<\pi. \quad (16)$$

Rewriting the expression (15) as

$$J=\frac{1}{y_1[y_2a+(1-y_2)b+\mathrm{i}\epsilon]}$$
$$\times\left(1+\frac{1-y_1}{y_1}\frac{y_2(1-y_2)c+\mathrm{i}\epsilon}{y_2a+(1-y_2)b+\mathrm{i}\epsilon}\right)^{-1}, \quad (17)$$

and utilizing the relation (16), J can be written in the form

$$J=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{d}s\frac{\Gamma(s)\Gamma(1-s)y_1^{s-1}}{[(1-y_1)(y_2(1-y_2)c+\mathrm{i}\epsilon)]^s}$$
$$\times[y_2a+(1-y_2)b+\mathrm{i}\epsilon]^{s-1}, \quad 0<\gamma<1. \quad (18)$$

Now, applying the relation (16) to the factor $(y_2a+(1-y_2)b+\mathrm{i}\epsilon)^{s-1}$, appearing on the right-hand side of (18), one finds that

$$J=\frac{1}{(2\pi\mathrm{i})^2}\frac{1}{a+\mathrm{i}\epsilon}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{d}s\int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty}\mathrm{d}s'$$
$$\times\Gamma(s)\Gamma(s')\Gamma(1-s-s')$$
$$\times\frac{y_1^{s-1}y_2^{s'-1}}{(1-y_1)^s(1-y_2)^{s+s'}}\left(\frac{c+\mathrm{i}\epsilon}{a+\mathrm{i}\epsilon}\right)^{-s}\left(\frac{b+\mathrm{i}\epsilon}{a+\mathrm{i}\epsilon}\right)^{-s'},$$
$$0<\gamma,\gamma'<1, \quad \gamma+\gamma'<1, \quad (19)$$

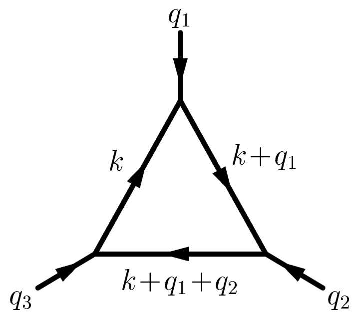

Fig. 2. One-loop triangle diagram

which represents the desired Mellin–Barnes representation for J given in (15). If use is made of the relation (19), the integral (14) becomes

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2)=\frac{\mathrm{i}}{(4\pi)^2}\frac{1}{(2\pi\mathrm{i})^2}\frac{1}{m_2^2+\mathrm{i}\epsilon} \quad (20)$$
$$\times\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{d}s\int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty}\mathrm{d}s'\Gamma(s)\Gamma(s')\Gamma(1-s-s')$$
$$\times\left(\frac{m_3^2+\mathrm{i}\epsilon}{m_2^2+\mathrm{i}\epsilon}\right)^{-s}\left(\frac{t+\mathrm{i}\epsilon}{m_2^2+\mathrm{i}\epsilon}\right)^{-s'}\int_0^1\mathrm{d}x_2\int_0^1\mathrm{d}x_3$$
$$\times\frac{x_2^{s-1}(1-x_2)^{-s}x_3^{s'-1}(1-x_3)^{-s-s'}}{x_2m_1^2+(1-x_2)x_3s+(1-x_2)(1-x_3)m_4^2+\mathrm{i}\epsilon}.$$

After performing the x_2 and x_3 integrations with the help of the formula

$$\int_0^1\mathrm{d}xx^{b-1}(1-x)^{a-b-1}(1-xz)^{-a}=\frac{\Gamma(b)\Gamma(a-b)}{\Gamma(a)(1-z)^b},$$
$$\mathrm{Re}a>\mathrm{Re}b>0, \quad |\arg(1-z)|<\pi, \quad (21)$$

one arrives at

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2)$$
$$=\frac{\mathrm{i}}{(4\pi)^2}\frac{1}{(2\pi\mathrm{i})^2}\frac{1}{(m_2^2+\mathrm{i}\epsilon)(m_4^2+\mathrm{i}\epsilon)}$$
$$\times\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{d}s\int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty}\mathrm{d}s'\,[\Gamma(s)\Gamma(s')\Gamma(1-s-s')]^2$$
$$\times\frac{(m_1^2+\mathrm{i}\epsilon)^{-s}}{(m_2^2+\mathrm{i}\epsilon)^{-s}}\frac{(m_3^2+\mathrm{i}\epsilon)^{-s}}{(m_4^2+\mathrm{i}\epsilon)^{-s}}\frac{(s+\mathrm{i}\epsilon)^{-s'}}{(m_2^2+\mathrm{i}\epsilon)^{-s'}}\frac{(t+\mathrm{i}\epsilon)^{-s'}}{(m_4^2+\mathrm{i}\epsilon)^{-s'}},$$
$$0<\gamma,\gamma'<1, \quad \gamma+\gamma'<1, \quad (22)$$

which is the sought Mellin–Barnes representation of the IR finite box scalar integral.

Let us now make a digression and, for a moment, consider the IR finite scalar one-loop triangle integral with massless internal lines in 4-dimensional space-time. It corresponds to the Feynman diagram of Fig. 2, and is given by

$$I_3(q_1,q_2,q_3)= \quad (23)$$
$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[k^2+\mathrm{i}\epsilon][(k+q_1)^2+\mathrm{i}\epsilon][(k+q_1+q_2)^2+\mathrm{i}\epsilon]},$$

where q_i, $i = 1, 2, 3$ are the incoming external momenta. Upon combining the denominators with the help of the Feynman parametrization formula, integrating out the loop momentum and introducing the external "masses" $q_i^2 = \mu_i^2 (i = 1, 2, 3)$, the integral (23) takes the form

$$I_3(\mu_1^2, \mu_2^2, \mu_3^2) = \frac{\mathrm{i}}{(4\pi)^2} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \delta(x_1 + x_2 + x_3 - 1) \times (x_1 x_2\, \mu_1^2 + x_2 x_3\, \mu_2^2 + x_1 x_3\, \mu_3^2 + \mathrm{i}\epsilon)^{-1}. \quad (24)$$

In this form it is evident that the triangle integral is invariant under arbitrary permutations of the external masses μ_i^2.

After eliminating the δ-function by performing the x_3 integration, introducing the new variable $x_2 \to (1 - x_1)x_2$ and taking into account (19), one can write

$$I_3(\mu_1^2, \mu_2^2, \mu_3^2) \quad (25)$$
$$= \frac{\mathrm{i}}{(4\pi)^2} \frac{1}{(2\pi\mathrm{i})^2} \frac{1}{\mu_1^2 + \mathrm{i}\epsilon} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{d}s \int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty} \mathrm{d}s' \times \Gamma(s)\Gamma(s')\Gamma(1-s-s') \left(\frac{\mu_2^2 + \mathrm{i}\epsilon}{\mu_1^2 + \mathrm{i}\epsilon}\right)^{-s} \left(\frac{\mu_3^2 + \mathrm{i}\epsilon}{\mu_1^2 + \mathrm{i}\epsilon}\right)^{-s'} \times \int_0^1 \mathrm{d}x_1 x_1^{s-1}(1-x_1)^{-s} \int_0^1 \mathrm{d}x_2 x_2^{s'-1}(1-x_2)^{-s-s'}.$$

Next, performing the x_1 and x_2 integrations with the help of the formula (21) (with $z = 0$), one obtains

$$I_3(\mu_1^2, \mu_2^2, \mu_3^2) = \frac{\mathrm{i}}{(4\pi)^2} \frac{1}{(2\pi\mathrm{i})^2} \frac{1}{\mu_1^2 + \mathrm{i}\epsilon} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{d}s \int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty} \mathrm{d}s' \times [\Gamma(s)\Gamma(s')\Gamma(1-s-s')]^2 \frac{(\mu_2^2 + \mathrm{i}\epsilon)^{-s}}{(\mu_1^2 + \mathrm{i}\epsilon)^{-s}} \frac{(\mu_3^2 + \mathrm{i}\epsilon)^{-s'}}{(\mu_1^2 + \mathrm{i}\epsilon)^{-s'}},$$
$$0 < \gamma, \gamma' < 1, \quad \gamma + \gamma' < 1, \quad (26)$$

which represents the Mellin–Barnes representation of the triangle scalar integral.

If we now compare the Mellin–Barnes representations of the scalar box and triangle integrals given by (22) and (26), respectively, we arrive at the relation

$$I_4(s, t; m_1^2, m_2^2, m_3^2, m_4^2) = I_3(st, m_1^2 m_3^2, m_2^2 m_4^2), \quad (27)$$

which is valid provided the following condition is satisfied:

$$(u + \mathrm{i}\epsilon)^{-s''}(v + \mathrm{i}\epsilon)^{-s''} = (uv + \mathrm{i}\epsilon)^{-s''}, \quad (28)$$

where (u, v) is any element of the set of ordered pairs in (11) and s'' stands for s or s'. As is easily seen, the above condition does not hold only if u and v are both negative $(u, v < 0)$. Since, as mentioned earlier, we are considering the cases with more positive than negative parameters, it is clear that the case when (28) is not valid can appear at most in one of the pairs.

Let us now examine in detail the product appearing on the left-hand side in (28), corresponding to the case when $u, v < 0$. One can then write

$$(u + \mathrm{i}\epsilon)^{-s''}(v + \mathrm{i}\epsilon)^{-s''} = |u|^{-s''}|v|^{-s''} \exp[-2\mathrm{i}\pi \mathrm{sign}(\epsilon) s'']$$
$$= (uv)^{-s''}[\cos(\pi s'') - \mathrm{i}\mathrm{sign}(\epsilon)\sin(\pi s'')]^2$$
$$= (uv)^{-s''}[1 - 2\mathrm{i}\mathrm{sign}(\epsilon)\sin(\pi s'')\cos(\pi s'') - 2\sin^2(\pi s'')]$$
$$= (uv)^{-s''}\{1 - 2\mathrm{i}\mathrm{sign}(\epsilon)\sin(\pi s'')\exp[-\mathrm{i}\pi \mathrm{sign}(\epsilon) s'']\}$$
$$= (uv + \mathrm{i}\epsilon)^{-s''} - 2\mathrm{i}\mathrm{sign}(\epsilon)\sin(\pi s'')(-uv + \mathrm{i}\epsilon)^{-s''}. \quad (29)$$

Next, with the help of the formula

$$\Gamma(z)\Gamma(1-z) = \pi \sin^{-1}(\pi z),$$

one finds that

$$(u + \mathrm{i}\epsilon)^{-s''}(v + \mathrm{i}\epsilon)^{-s''} = (uv + \mathrm{i}\epsilon)^{-s''} - \frac{2\mathrm{i}\pi \mathrm{sign}(\epsilon)}{\Gamma(s'')\Gamma(1-s'')}(-uv + \mathrm{i}\epsilon)^{-s''}. \quad (30)$$

Owing to the symmetry of the pairs, with no loss of generality, we can now suppose that the "problematic pair" (i.e. the pair in which both masses are negative) is (m_1^2, m_3^2). In this case, the first term on the right-hand side in (30) is of the form that makes it possible to represent the box integral in terms of the triangle integral, while the second term is a correction originating from the fact that the first term does not keep the information regarding the signs of u and v.

Let us now evaluate the correction term. Upon combining (22) and (30), the Mellin–Barnes representation of the correction term is

$$K(s, t, m_1^2, m_3^2, m_2^2, m_4^2) = \frac{\mathrm{i}}{(4\pi)^2} \frac{1}{(2\pi\mathrm{i})^2} \frac{-2\mathrm{i}\pi \mathrm{sign}(\epsilon)}{m_2^2 m_4^2 + \mathrm{i}\epsilon} \times \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{d}s \int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty} \mathrm{d}s' \frac{\Gamma(s)\,[\Gamma(s')\Gamma(1-s-s')]^2}{\Gamma(1-s)} \times \frac{(-m_1^2 m_3^2 + \mathrm{i}\epsilon)^{-s}}{(m_2^2 m_4^2 + \mathrm{i}\epsilon)^{-s}} \frac{(st + \mathrm{i}\epsilon)^{-s'}}{(m_2^2 m_4^2 + \mathrm{i}\epsilon)^{-s'}}, \quad (31)$$

which, after making use of the formula (21) (with $z = 0$, $a = 1 - s$ and $b = s'$), is found to take the form

$$K(s, t, m_1^2, m_3^2, m_2^2, m_4^2) = \frac{\mathrm{i}}{(4\pi)^2} \frac{1}{(2\pi\mathrm{i})^2} \frac{-2\mathrm{i}\pi \mathrm{sign}(\epsilon)}{m_2^2 m_4^2 + \mathrm{i}\epsilon} \times \int_0^1 \mathrm{d}y \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{d}s \int_{\gamma'-\mathrm{i}\infty}^{\gamma'+\mathrm{i}\infty} \mathrm{d}s' \Gamma(s)\Gamma(s')\Gamma(1-s-s') \times \frac{y^{s'-1}}{(1-y)^{s+s'}} \frac{(-m_1^2 m_3^2 + \mathrm{i}\epsilon)^{-s}}{(m_2^2 m_4^2 + \mathrm{i}\epsilon)^{-s}} \frac{(st + \mathrm{i}\epsilon)^{-s'}}{(m_2^2 m_4^2 + \mathrm{i}\epsilon)^{-s'}}. \quad (32)$$

If we now compare this expression with the expressions (15) and (19), with $y_1 = 1/2$, we get

$$K(s, t, m_1^2, m_3^2, m_2^2, m_4^2) = -2\mathrm{i}\pi \mathrm{sign}(\epsilon) \frac{\mathrm{i}}{(4\pi)^2} \quad (33)$$
$$\times \int_0^1 \mathrm{d}y \big[y m_2^2 m_4^2 + (1-y)st - y(1-y) m_1^2 m_3^2 + \mathrm{i}\epsilon\big]^{-1}.$$

Next, after employing the identity

$$\frac{1}{ym_2^2m_4^2+(1-y)st-y(1-y)m_1^2m_3^2+\mathrm{i}\epsilon} = \frac{1}{\lambda}\left[\frac{1}{y-(y_1-\mathrm{i}\epsilon)}-\frac{1}{y-(y_2+\mathrm{i}\epsilon)}\right], \quad (34)$$

where

$$y_1=\frac{m_1^2m_3^2-m_2^2m_4^2+st+\lambda}{2m_1^2m_3^2}$$

and

$$y_2=\frac{m_1^2m_3^2-m_2^2m_4^2+st-\lambda}{2m_1^2m_3^2}$$

are the roots of the equation

$$ym_2^2m_4^2+(1-y)st-y(1-y)m_1^2m_3^2=0,$$

while

$$\lambda=\left[(st)^2+(m_1^2m_3^2)^2+(m_2^2m_4^2)^2-2stm_1^2m_3^2 - 2stm_2^2m_4^2-2m_1^2m_3^2m_2^2m_4^2\right]^{1/2}, \quad (35)$$

and performing the remaining integration, we arrive at the final result for the correction term:

$$K(s,t,m_1^2,m_3^2,m_2^2,m_4^2)=-2\mathrm{i}\pi\,\mathrm{sign}(\epsilon)\frac{\mathrm{i}}{(4\pi)^2\lambda} \times\left[\ln\left(m_2^2m_4^2+st-(m_1^2m_3^2-\lambda)(1+\mathrm{i}\epsilon)\right) -\ln\left(m_2^2m_4^2+st-(m_1^2m_3^2+\lambda)(1-\mathrm{i}\epsilon)\right)\right]. \quad (36)$$

As is seen, the correction term K is written in the form in which the symmetry with respect to the pairs (s,t) and (m_2^2,m_4^2) is evident.

Based on the above consideration, the expression for the IR finite one-loop scalar box integral can be written as

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2,\epsilon)=I_3(st,m_1^2m_3^2,m_2^2m_4^2,\epsilon) +K(s,t,m_1^2,m_3^2,m_2^2,m_4^2,\epsilon). \quad (37)$$

The first term on the right-hand side in (37) represents the expression for the IR finite triangle scalar integral. It is evaluated in detail in the appendix, and is given by

$$I_3(\mu_1^2,\mu_2^2,\mu_3^2,\epsilon) =\frac{\mathrm{i}}{(4\pi)^2}\frac{1}{\nu_3(x_1-x_2)}\left\{2\mathrm{Li}_2\left(\frac{1}{x_2}\right) -2\mathrm{Li}_2\left(\frac{1}{x_1}\right)+\ln\left[x_1x_2+\mathrm{i}\epsilon\,\mathrm{sign}(\nu_3)\right] \times\left[\ln\left(\frac{1-x_1}{-x_1}\right)-\ln\left(\frac{1-x_2}{-x_2}\right)\right]\right\}, \quad (38)$$

where x_1 and x_2 are the roots of the equation

$$x\nu_1+(1-x)\nu_2-x(1-x)\nu_3=0,$$

and

$$\{\nu_1,\nu_2,\nu_3\}=\mathcal{P}\{\mu_1^2,\mu_2^2,\mu_3^2\},$$

where $\mathcal{P}$ denotes the permutation of the masses chosen in such a way that the roots x_1 and x_2 are outside the interval $[0,1]$. If the signs of all the masses μ_i^2 are the same, ν_3 should be chosen so as to correspond to the mass with the smallest absolute value. On the other hand, if the signs of the masses are not all the same, ν_3 ought to be chosen so as to coincide with the mass whose sign is opposite to the sign of the other two masses.

The result (38) is similar to the result given in [11] but is more convenient for numerical calculations, since it is easier to separate the real and imaginary part. Note that as a consequence of choosing ν_i in an appropriate way, the causal iϵ appears only in one of the logarithms in (38).

The second term on the right-hand side in (37) is the correction term. As stated earlier, it accounts for the fact that in the product of two kinematic variables we lose information about the phases which can lead to missing some poles. The expression for the correction term given in (36) corresponds to a special case characterized by the fact that the $m_1^2,m_3^2<0$. This term, however, can be written in a general and for practical purposes more convenient form:

$$K(\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\epsilon) =\frac{\mathrm{i}}{(4\pi)^2}\frac{-2\mathrm{i}\pi\,\mathrm{sign}(\epsilon)}{\lambda}\sum_{i=1}^{3}\theta(-\alpha_i)\theta(-\beta_i) \times\left[\ln\left(\sum\nolimits_{j\neq i}\alpha_j\beta_j-(\alpha_i\beta_i-\lambda)(1+\mathrm{i}\epsilon)\right) -\ln\left(\sum\nolimits_{j\neq i}\alpha_j\beta_j-(\alpha_i\beta_i+\lambda)(1-\mathrm{i}\epsilon)\right)\right], \quad (39)$$

with λ given in (35). It should be observed that in practice at most one of the terms in the sum in (39) is different from zero.

The above result for the IR finite scalar box integral, as stated earlier, has been derived under the assumption that the number of positive kinematic variables is larger than the number of negative ones. If the opposite is true, then owing to the symmetry

$$I_4(s,t;m_1^2,m_2^2,m_3^2,m_4^2,\epsilon) =I_4(-s,-t;-m_1^2,-m_2^2,-m_3^2,-m_4^2,-\epsilon), \quad (40)$$

which is obvious from (10), the corresponding expression can simply be obtained from the one given above simply by changing the signs of all six kinematic variables as well as of the causal ϵ.

Finally, we have numerically checked our result for the IR finite scalar box integral with massless internal lines with the corresponding result of [9] and found agreement in all the kinematic regions.

4 Conclusion

Using the Mellin–Barnes integral representations, the calculation of the IR finite one-loop box scalar integral with massless internal lines has been reduced to the calculation of the IR finite triangle scalar integral with massless internal lines. The result is presented in a very simple and compact form (only two dilogarithms) and, owing to the fact that we have kept the causal iϵ systematically throughout the calculation, is quite general, i.e. valid for arbitrary values of the relevant kinematic variables. From the practical point of view, the main advantage of our result over the results previously obtained is that the arguments of the occurring logarithms and dilogarithms are very simple, which enables one to separate the real and imaginary part of the integral very easily. This makes our result for the IR finite one-loop scalar box integral more appropriate for numerical calculations.

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Appendix

In this appendix we present the details of the calculation of the IR massless scalar one-loop triangle integral given in (38).

Eliminating the δ-function in (24) by performing the x_3 integration, making the change of variables $x_2 \to (1-x_1)x_2$ and performing the x_1 integration, the integral (24) becomes

$$I_3(\mu_1^2,\mu_2^2,\mu_3^2) = \frac{\mathrm{i}}{(4\pi)^2}\int_0^1 \mathrm{d}x \times \frac{\ln\left[x\mu_1^2+(1-x)\mu_3^2+\mathrm{i}\epsilon\right]-\ln\left[x(1-x)\mu_2^2+\mathrm{i}\epsilon\right]}{x\mu_1^2+(1-x)\mu_3^2-x(1-x)\mu_2^2}, \quad (41)$$

where we have replaced the variable x_2 by x. The poles of the integrand are

$$x_{1,2} = \frac{1}{2}\left[1-\frac{\mu_1^2}{\mu_2^2}+\frac{\mu_3^2}{\mu_2^2} \pm \sqrt{\left(1-\frac{\mu_1^2}{\mu_2^2}-\frac{\mu_3^2}{\mu_2^2}\right)^2 - 4\frac{\mu_1^2}{\mu_2^2}\frac{\mu_3^2}{\mu_2^2}}\right]. \quad (42)$$

As can easily be seen from the denominator in (41), the integration path will not pass through the pole if one of the following two cases is realized: first, if the sign of μ_2^2 is opposite to the signs of μ_1^2 and μ_3^2; second, if all the masses are of the same sign and μ_2^2 has the smallest absolute value. Now, owing to the fact that the integral $I_3(\mu_1^2,\mu_2^2,\mu_3^2)$ is invariant under the permutations of the parameters μ_i^2, one can easily show that it is always possible to permute the parameters μ_i^2 in a way that the poles x_1 and x_2 are outside the interval $[0,1]$. In trying to accomplish that, one of the following four possibilities can arise: the poles are complex conjugate, both poles are in the interval $\langle -\infty, 0\rangle$, both poles are in the interval $\langle 1,\infty\rangle$ and one pole is in the interval $\langle -\infty,0\rangle$ and another in the interval $\langle 1,\infty\rangle$. In the following considerations we assume that the parameters μ_i^2 are already chosen in such a way that the above requirement is satisfied.

Due to the fact that the poles belong to one of the above-mentioned cases, it is convenient to rewrite the integral (41) in terms of x_1 and x_2. The integral (41) then takes the form

$$\begin{aligned} I_3(\mu_1^2,\mu_2^2,\mu_3^2) &= \frac{\mathrm{i}}{(4\pi)^2}\frac{1}{\mu_2^2}\int_0^1 \mathrm{d}x\frac{1}{(x-x_1)(x-x_2)} \\ &\times \left(\ln\left[1+x\frac{1-x_1-x_2}{x_1x_2}\right]-\ln\left[x(1-x)\right]\right. \\ &\left. +\ln\left[x_1x_2+\mathrm{i}\epsilon\,\mathrm{sign}(\mu_2^2)\right]\right). \end{aligned} \quad (43)$$

Observe that the arguments of the first two logarithms are positive and that there are no poles on the path of the integration. Now, by the partial fraction decomposition the integral (43) can be written as

$$\begin{aligned} I_3(\mu_1^2,\mu_2^2,\mu_3^2) &= \frac{\mathrm{i}}{(4\pi)^2}\frac{1}{\mu_2^2(x_1-x_2)} \\ &\times\left[G_1(x_1,x_2)+G_2(x_1,x_2)+G_3(x_1,x_2)\right], \end{aligned} \quad (44)$$

where

$$\begin{aligned} G_1(x_1,x_2) &= \int_0^1 \mathrm{d}x\frac{1}{x-x_1}\ln\left[1+x\frac{1-x_1-x_2}{x_1x_2}\right] \\ &- \int_0^1 \mathrm{d}x\frac{1}{x-x_2}\ln\left[1+x\frac{1-x_1-x_2}{x_1x_2}\right], \end{aligned} \quad (45)$$

$$\begin{aligned} G_2(x_1,x_2) &= -\int_0^1 \mathrm{d}x\frac{1}{x-x_1}\ln\left[x(1-x)\right] \\ &+ \int_0^1 \mathrm{d}x\frac{1}{x-x_2}\ln\left[x(1-x)\right], \end{aligned} \quad (46)$$

$$\begin{aligned} G_3(x_1,x_2) &= \ln\left[x_1x_2+\mathrm{i}\epsilon\,\mathrm{sign}(\mu_2^2)\right] \\ &\times\left(\int_0^1 \mathrm{d}x\frac{1}{x-x_1} - \int_0^1 \mathrm{d}x\frac{1}{x-x_2}\right). \end{aligned} \quad (47)$$

Let us now evaluate the above integrals in turn. Changing the integration variable

$$1+x\frac{1-x_1-x_2}{x_1x_2} \to \frac{1-x_1}{x_2}x$$

in the first, and

$$1+x\frac{1-x_1-x_2}{x_1x_2} \to \frac{1-x_2}{x_1}x$$

in the second integral in (45), one finds

$$\begin{aligned} G_1(x_1,x_2) &= \int_{x_2/(1-x_1)}^{(1-x_2)/x_1} \mathrm{d}x\frac{1}{x-1}\ln\left[\frac{1-x_1}{x_2}x\right] \\ &- \int_{x_1/(1-x_2)}^{(1-x_1)/x_2} \mathrm{d}x\frac{1}{x-1}\ln\left[\frac{1-x_2}{x_1}x\right]. \end{aligned} \quad (48)$$

Next, after passing to the new integration variable $x \to 1/x$ in the second integral above, making a small rearrangement and performing two elementary integrations, the final expression for the integral (45) turns out to be

$$G_1(x_1, x_2) = \frac{1}{2}\ln^2\left(\frac{1-x_1}{-x_1}\right) - \frac{1}{2}\ln^2\left(\frac{1-x_2}{-x_2}\right). \quad (49)$$

As for the integral (46), one can write it in the form

$$G_2(x_1, x_2) = -\int_0^1 \mathrm{d}x \frac{\ln(x)}{x - x_1} + \int_0^1 \mathrm{d}x \frac{\ln(x)}{x-(1-x_1)} + \int_0^1 \mathrm{d}x \frac{\ln(x)}{x - x_2} - \int_0^1 \mathrm{d}x \frac{\ln(x)}{x-(1-x_2)}, \quad (50)$$

which, when expressed in terms of the Euler dilogarithms, takes the form

$$G_2(x_1, x_2) = -\mathrm{Li}_2\left(\frac{1}{x_1}\right) + \mathrm{Li}_2\left(\frac{1}{1-x_1}\right) + \mathrm{Li}_2\left(\frac{1}{x_2}\right) - \mathrm{Li}_2\left(\frac{1}{1-x_2}\right). \quad (51)$$

Next, applying the identity

$$\mathrm{Li}_2\left(\frac{1}{1-z}\right) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\ln^2\left(\frac{1-z}{-z}\right)$$

in (51), we arrive at

$$G_2(x_1, x_2) = -2\mathrm{Li}_2\left(\frac{1}{x_1}\right) + 2\mathrm{Li}_2\left(\frac{1}{x_2}\right) - \frac{1}{2}\ln^2\left(\frac{1-x_1}{-x_1}\right) + \frac{1}{2}\ln^2\left(\frac{1-x_2}{-x_2}\right). \quad (52)$$

The remaining integral (47) is elementary and given by

$$G_3(x_1, x_2) = \ln\left(x_1 x_2 + \mathrm{i}\epsilon \,\mathrm{sign}(\mu_2^2)\right) \times \left[\ln\left(\frac{1-x_1}{-x_1}\right) - \ln\left(\frac{1-x_2}{-x_2}\right)\right]. \quad (53)$$

Upon combining (44), (49), (52) and (53), we arrive at the final expression for the IR finite triangle scalar integral given by (38).

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